Scale-dependent dimension in the forest fire model

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The forest fire model is a reaction-diffusion model where energy, in the form of trees, is injected uniformly, and burned (dissipated) locally. We show that the spatial distribution of fires forms a geometric structure where the fractal dimension varies continuously with the length scale. In the three-dimensional model, the dimensions vary from zero to three, proportional with $\ln(l)$, as the length scale increases from $l\sim 1$ to a correlation length $l=\xi$. Beyond the correlation length, which diverges with the growth rate p as $\xi\propto p^{-2/3}$, the distribution becomes homogeneous. We suggest that this picture applies to the ''intermediate range'' of turbulence where it provides a natural interpretation of the extended scaling that has been observed at small length scales. Unexpectedly, it might also be applicable to the spatial distribution of luminous matter in the universe. In the two-dimensional version, the dimension increases to $D=1$ at a length scale $l\sim 1/p$, where there is a crossover to homogeneity, i.e., a jump from $D=1$ to $D=2$.

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I. INTRODUCTION

Systems undergoing continuous phase transitions are usually described in terms of a correlation length ξ that diverges as the critical point is approached. For length scales below the correlation length, the geometrical structure is self-similar, with a unique fractal dimension that is independent of l for $l\ll\xi$. For larger length scales the structure is uniform. Self-similarity can be conveniently described as a fixed point of a renormalization-group transformation. This behavior includes self-organized critical phenomena, where the correlation length diverges as a powerlaw when the driving rate vanishes.

We propose a geometric form for nonequilibrium systems, where the dimension of the dissipative field varies gradually from $D=0$ at the smallest scale, dominated by point-like objects, to $D=3$, or bulklike, at some finite correlation length ξ. The density remains uniform for length scales exceeding ξ. Thus, as one steps further and further backwards, and consider things at a larger and larger length scale l, different classes of fractal objects are observed, ranging from points to lines to walls, and finally to a homogeneous set.

The new form of scaling appears in our study of the simple forest fire model (FFM) [1], which was proposed in an attempt to throw light on the nature of the spatial distribution of energy dissipation in fully developed turbulence. It is a discrete model defined on a lattice in the best traditions of Ising-like models used to study equilibrium phase transitions. In turbulence, energy is injected at a large scale, and dissipated at the smallest scale. In our vision, the forest fires would represent the intermittency observed in turbulence, and power-law spatial and temporal correlations would naturally occur if the model operates at the self-organized critical state.

It turns out that, while the model indeed has a correlation length that diverges as the growth rate p goes to zero, there is no fractal self-similarity below the correlation length, as is usually the case for critical systems, self-organized or not. The criticality can therefore not be described in terms of a fixed point in the Wilson sense. For the three-dimensional version, we find that the length-scale-dependent dimension $D(l)$ (for $l<\xi$) is given by a remarkably simple equation,

$$D(l)\sim 3\ln(l/l_0)/\ln(\xi/l_0). \qquad (1)$$

where l_0 is a length of order unity. In two-dimensions, the exponent increases gradually to $D=1$ at the correlation length, where there is a normal crossover to two-dimensional homogeneity. Whether or not one would actually call this ''critical'' behavior is a matter of taste.

While this geometric structure was not what we were looking for, it may nevertheless turn out to represent the actual scaling in turbulence. In the scaling regime, or inertial range, the energy dissipation field in turbulence is known to be homogeneous with great accuracy. However, there appears to be an ''intermediate range'' [2], rather than a single lower cutoff length, where the interesting intermittent behavior that we usually associate with turbulence takes place. The correlations do not follow power laws, but are characterized by smoothly varying effective exponents. Only the relative moments follow scale-independent ratios. This can perhaps be interpreted as a scale-dependent exponent for the dimension of the dissipative field, as observed in our model.

On a quite different front, there has been much controversy about the spatial distribution of luminous matter in the universe. The apparent hierarchical structure has led many researchers to believe that the distribution could be self-similar, or fractal [3]. However, the uniformity of the background radiation requires that the universe is homogeneous at the largest scale, contradicting the simple self-similar scaling picture. Perhaps the luminous matter in the universe obeys this type of geometry, which unifies both observations. While we hesitate to claim that the universe should be viewed as one giant forest fire, we do suggest that the scaling picture may represent a quite general geometrical form for nonequilibrium dissipative systems.

II. NUMERICAL RESULTS

The forest fire model is defined as follows. On a d-dimensional lattice, trees are grown randomly at a rate p. During a time unit, trees burn down (leaving room for new trees), and ignite their neighbors. After a transient period, the system enters a statistically stationary state with a complex distribution of fires (and forests). This state is the object of our investigation.

Despite the glaring simplicity of the FFM, reaching an understanding of its properties has turned out to be an elusive goal. Grassberger and Kantz [4] argued that there could be no criticality since the dynamics is simply that of domain walls moving with a finite mean velocity, burning everything in their wake. The motion of the walls gives rise to local oscillations in the fire intensity. This view was more or less accepted by the community, and interest in the model dropped. In the mean time, Drossel and Schwabl [5] invented a different version of the FFM where fires are injected at a small but finite rate, which exhibits noncontroversial conventional self-organized criticality [6]. It can be exactly solved in one dimension [7,8] in terms of a cascade process.

However, a few years ago, Johansen [9] found that the time between two fronts is not limited by the growth rate, since the fire walls can propagate without burning much of the forest. In fact, the fraction of trees that burn vanishes as $p \rightarrow 0$. The process should be seen as a percolationlike process rather than as solid fire fronts. Soon after, Broker and Grassberger [10] confirmed that the periodicity of local oscillations in the fire density diverges with a nontrivial exponent, as $p \rightarrow 0$.

Our picture of the $D=3$ FFM does not involve the concept of walls or spirals: they only exist as well-defined quantities in the mind of the observer when the system is viewed at a particular scale. In two dimensions, walls are still meaningful, representing the largest coherent structures.

We simulated both two- and three-dimensional systems with sizes up to 2048^2 and 1024^3, respectively. 0.5×10^6 time steps were used to collect the statistics for each simulation. Special care is needed for the simulation of a sustained forest fire due to the fact that for a given system size L, the fire dies out if the rate is not sufficiently high. This feature turns out to be important for applications to the spread of diseases, and to the existence of luminous matter in the universe. If artificially restarted (say, by just adding a single fire), the system often goes into a state with global oscillations. Typically we can only study system sizes that are much larger than the correlation length. This contrasts with conventional critical phenomena, where information for length scales up to the size of the system can be obtained even for systems that are smaller than the correlation length. Finite-size scaling allows one to derive critical exponents from studying such systems. Here, in contrast, the entire dynamics collapses as soon as the correlation length reaches the system size. Thus, the range of scales l that we can study is squeezed by the inequalities $1 < l < \xi \ll L$, making the calculation numerically demanding despite the simplicity of the model. Our simulations involved more than 10^9 sites.

The average amount of fires $n(l)$ within boxes of size l that contain fires was measured. Figure 1(a) shows $\ln(n)$ vs $\ln(l)$ for a wide range of p and $L=128,256,516,1024$ for d

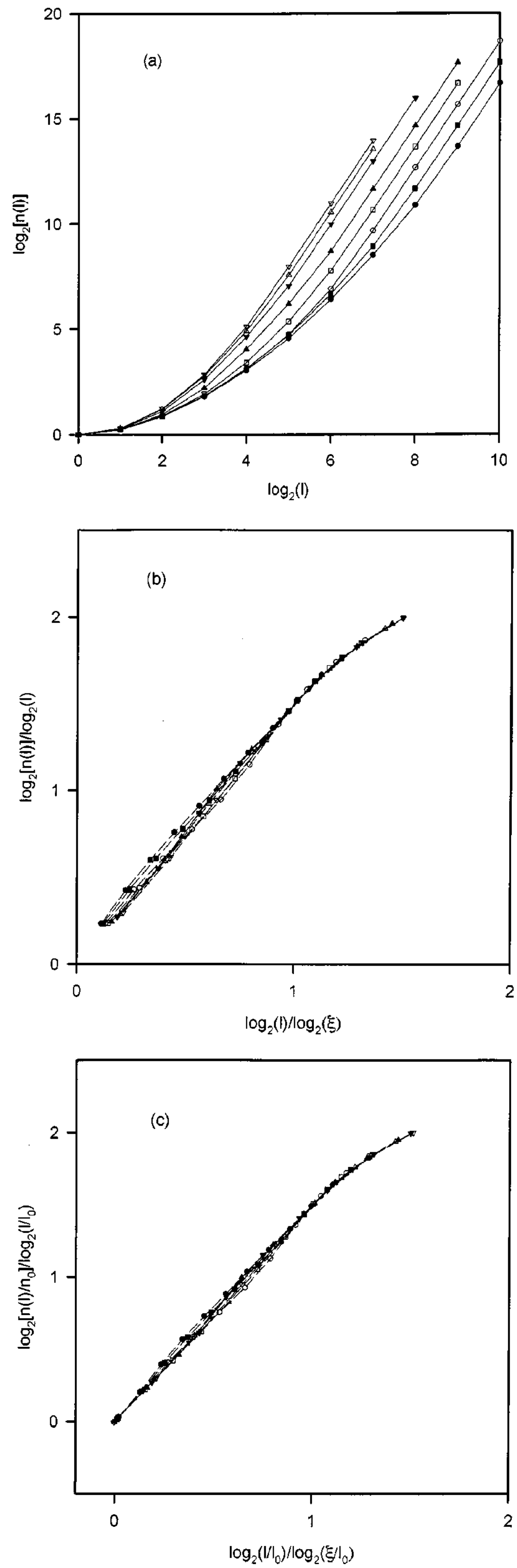

FIG. 1. (a) $\log_2[n(l)]$ vs $\log_2(l)$ for various system sizes and tree growth rates p for the 3D forest fire model: filled circle ($1024^3, p=1.25\times 10^{-4}$), filled square ($1024^3, p=2.5\times 10^{-4}$), open circle ($1024^3, p=5\times 10^{-4}$), open square ($512^3, p=0.001$), filled triangle ($512^3, p=0.002$), inverted filled triangle ($256^3, p=0.005$), open triangle ($128^3, p=0.0075$), inverted open triangle ($128^3, p=0.01$). (b) $\ln[n(l)]/\ln(l)$ vs $\ln(l)/\ln(\xi)$ with the same set of data. The correlation lengths are given by $\xi=(0.77p)^{-2/3}$. (c) $\ln[n(l)/n_0]/\ln(l/l_0)$ vs $\ln(l/l_0)/\ln(\xi/l_0)$, with the lower cutoff $l_0 \propto p^{0.03}$.

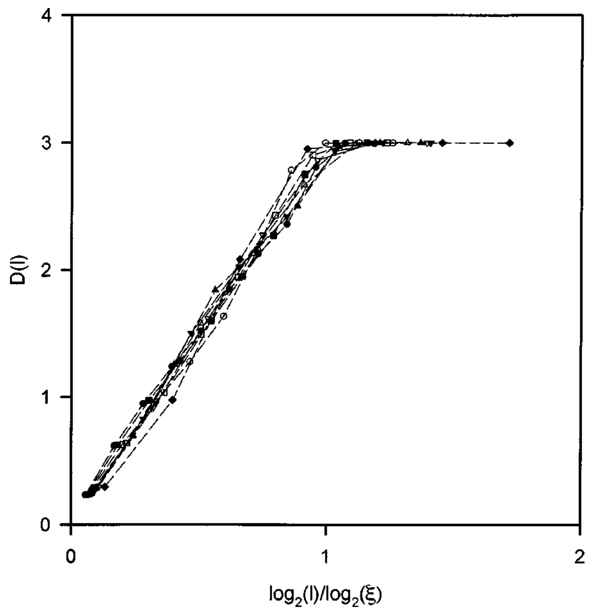

FIG. 2. The length-dependent fractal dimension $D(l) = d\ln[n(l)]/d\ln(l)$ (calculated from differences between the data points in Fig. 1) vs $\ln(l)/\ln(\xi)$ for the data sets used in Fig. 1, with $\xi=(0.77p)^{-2/3}$.

$=3$. There is no linear regime, indicating that there is no well-defined fractal dimension, in contrast to our original claim. The slopes of curves generally increase with l, and saturate at a value of 3 at larger l values, indicating that the distribution becomes uniform beyond a length scale that we identify as the correlation length. Also, n increases with p since the number of fires in average must be equal to the growth rate in the stationary state.

A unified description can be obtained by plotting $\ln(n)/\ln(l)$ vs $\ln(l)/\ln(\xi)$ for the same data [Fig. 1(b)], where good data collapse, involving length scales extending over three orders of magnitude, is obtained when choosing $\xi =(0.77p)^{-2/3}$ ($\nu=2/3$). Actually, a slightly better fit is obtained if the lower cutoff is allowed to depend on p, so that in principle there are two adjustable exponents. Fig. 1(c) shows $\ln(n/n_0)/\ln(l/l_0)$ vs $\ln(l/l_0)/\ln(\xi/l_0)$ with $l_0 \propto p^{0.03}$.

The data collapse implies that $\ln(n/n_0)/\ln(l/l_0)$ can be written in the form

$$\ln(n)/\ln(l/l_0) = f[\ln(l/l_0)/\ln(\xi/l_0)], \qquad (2)$$

where the collapsed curve represents the function f. It turns out to be instructive to define the derivative

$$D(l) = d\ln(n)/d\ln(l) = \alpha[\ln(l/l_0)/\ln(\xi/l_0)], \qquad (3)$$

which can be thought of as a length-scale-dependent fractal dimension. This quantity is shown in Fig. 2. The function α is given by $\alpha(x) = f(x) + x f'(x)$. The curve has an interesting and unusual shape: the function is linear for length scales up to the correlation length where there is a sharp kink. Beyond the correlation length, the value of d is 3: the system is homogeneous for length scales beyond the correlation length. The data collapse shown in Figs. 1(b,c) and 2 indicates bona fide scaling, in the sense that there is only one relevant length ξ/l_0 in the system, although of a quite novel and unique nature, without self-similarity under a change of scale.

Thus, we arrive at the extremely simple Eq. (1), which is our main result. This shows how the apparent dimension increases as the forest is viewed at larger and larger distances. At the smallest scales, the fires are zero dimensional and appear point like and isolated. As the scale increases, the dimension increases logarithmically until at the correlation length it becomes equal to three. It would be interesting to have a computer-generated graphical visualization of this change of dimension.

The amount of fires within a box of size l becomes

$$\ln(n) \sim \left(\frac{3}{2}\frac{\ln(l/l_0)}{\ln(\xi/l_0)}\right)\ln(l/l_0). \qquad (4)$$

The exponent ν can be derived analytically by an argument of energy conservation. The number of fires $n(\xi)$ in a box of size ξ times the number of boxes of that size, $(L/\xi)^d$, scales as pL^d. Since the fractal dimension $D(l)$ is linearly dependent on $\ln(l)$, we have $n(\xi)=\xi^{d/2}$ (ignoring the small p dependence of l_0); this leads to $\nu=2/d$.

It is interesting that the correlation function throughout the scaling region $l \ll \xi$ is influenced by both the correlation length and the smallest length scale of dissipation. For example, one can estimate the correlation length by measuring the increase of dimensionality from one small length scale to another. In contrast, for conventional critical phenomena, the properties up to the correlation length are those of the critical state, and for $l \ll \xi$ there is no way to detect ξ. In addition, the scaling form is invariant with respect to the transformation $l \to l^\gamma$, $\xi \to \xi^\gamma$, and $n(l) \to n(l)^\gamma$, which leaves the smallest dissipation scale (at $l \sim 1$) unchanged.

Similar data for $d=2$ are shown in Fig. 3. The dimension grows from a small value close to zero at low length scales, and then jumps from $D \sim 1$ to $D=2$ at the correlation length. The variation is nonlinear, but nevertheless there are good data collapses for $\nu=1$. For a range of length scales less than the correlation length, the dimension is close to 1, indicating that walls form the largest coherent structures in the two-dimensional forest fire model. Again, at length scales greater than the correlation length, the density becomes uniform, and $D=2$. There is no range of length scales where the dimension is between 1 and 2. We have also studied the higher moments of the fire distribution, and similar scaling forms were found. The picture of propagating fronts remain valid; the features seen at lower length scales represent the internal structure of the walls, which also scales with the correlation length.

The forest fire model was originally thought of as a toy model of turbulence. Recently, deviations from fractal or multifractal scaling have been interpreted as ‘‘extended self-similarity’’ in an intermediate dissipative range [2,11] between the Kolmogorov length and the inertial range. Perhaps one might understand this phenomenon geometrically in terms of the concept of scale-dependent dimension introduced here. In particular, data presented by Benzi *et al.* [12] seem to indicate a logarithmic dependence of scaling exponents versus length scale. For homogeneous turbulence, the energy dissipation scales with the third moment of the veloc-

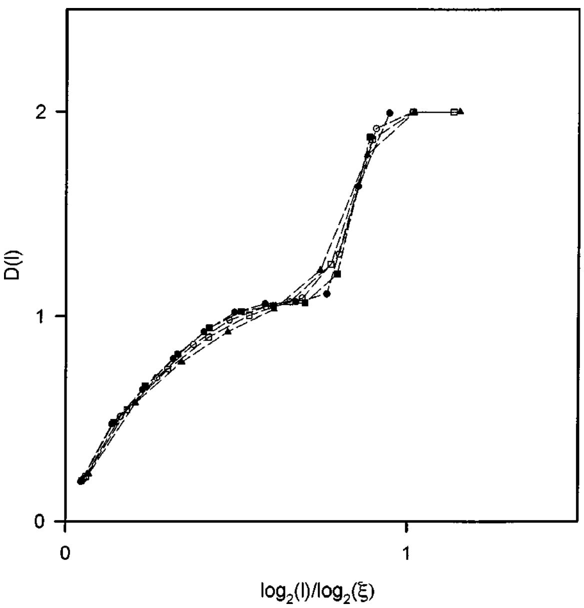

FIG. 3. The length-dependent fractal dimension $D(l)=d\ln[n(l)]/d\ln(l)$ vs $\ln(l)/\ln(\xi)$ for the two-dimensional forest fire model: filled circle $(2048^2, p=0.00075)$, filled square $(1024^2, p=0.001)$, open circle $(1024^2, p=0.0025)$, open square $(1024^3, p=0.005)$, filled triangle $(516^2, p=0.01)$. The correlation lengths used are given by $\xi=(0.60p)^{-1}$.

ity differences. It would be interesting to plot the deviations in the intermediate range in order to check whether it could be accounted for by a scale-dependent dimension as given by Eq. (1). Other experiments showing a dimension depending on the Reynolds number [13,14] might alternatively be interpreted as a scale-dependent dimension. In any case, the view of turbulence as a forest fire could constitute a powerful metaphorical picture.

It is important to distinguish our geometric structure from that of a random distribution of fires with the same density. Such a structure would produce a $D(l)$ vs $\ln(l)$ curve that would be zero until a length equal to the cube root of the density, i.e., the average length. Then there would be a crossover jump to $D=3$. Similarly, a fractal structure would show up as a constant $D(l)$ up to the correlation length; then there would be a jump to the Euclidean dimension.

The structure of the universe: Could it indeed be that the universe operates at a similar self-organized state with the spatial distribution of bright matter characterized by the logarithmic scaling? We have analyzed galaxy maps in order to test this proposal, with very good agreement [15]. The fit yields a correlation length of approximately 300 Mpc, which is outside the range of present galaxy catalogs, but will be reached within a decade. It would be exciting to check if the structure indeed is represented by the kinkcurve, Fig. 2.

So far, we have not reached an analytical understanding of this simple type of scaling. Traditional renormalization-group analysis based on rescaling of length scale will not apply here, and it is our belief that a quite different framework might be needed.

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